Non-genericity of the Nariai solutions:
I. Asymptotics and spatially homogeneous perturbations

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Abstract

This is the first of two papers where we study the asymptotics of the generalized Nariai solutions and its relation to the cosmic no-hair conjecture. According to the cosmic no-hair conjecture, generic expanding solutions of Einstein's field equations in vacuum with a positive cosmological constant isotropize and approach the de-Sitter solution asymptotically. The family of solutions which we introduce as "generalized Nariai solutions", however, shows quite unusual asymptotics and hence should be non-generic in some sense. In this paper, we list basic facts for the Nariai solutions and characterize their asymptotic behavior geometrically. One particular result is a rigorous proof of the fact that the Nariai solutions do not possess smooth conformal boundaries. We proceed by explaining the non-genericity within the class of spatially homogeneous solutions. It turns out that perturbations of the three isometry classes of generalized Nariai solutions are related to different mass regimes of Schwarzschild de-Sitter solutions. A motivation for our work here is to prepare the second paper devoted to the study of the instability of the Nariai solutions for Gowdy symmetry. We will be particularly interested in the construction of new and in principle arbitrarily complicated cosmological black hole solutions.

1 Introduction

The general knowledge of qualitative properties and phenomena for cosmological solutions of Einstein's field equations is still quite limited. Of particular fundamental importance are the strong cosmic censorship and the so-called BKL-conjectures. The reader can find relevant background material in [1]. Recently, there have been several break-throughs, but still many issues are open due to the complexity of the interplay of geometry and highly complicated partial differential equations in Einstein's theory. In this paper, another fundamental open problem will be the main concern. In a restricted setting, we will study the asymptotic behavior of rapidly expanding solutions of the field equations.

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A deeper understanding of such solutions is crucial for the correct interpretation of cosmological observations. The standard model of cosmology, whose details are explained extensively e.g. in [21], is surprisingly consistent with current observations [25, 26]. A keystone of the standard model is inflation in the very early universe, i.e. a period characterized by rapid expansion. The theory of inflation has mostly been formulated in the context of spatially homogeneous and isotropic models. However, strong inhomogeneities can be expected in the very early universe caused by quantum fluctuations and other physical processes which are not yet completely understood. One is interested in the problem of whether inflation is able to homogenize and isotropize the solutions despite possible strong inhomogeneities before inflation. If so, this would yield a natural explanation for the apparent homogeneity and isotropy of our universe without the necessity of ad-hoc assumptions about the beginning of the universe. Let us assume the simplest model for dark energy, namely a positive cosmological constant $\Lambda > 0$ in Einstein's field equations. Recall that dark energy is the name for that hypothetical matter component in our universe responsible for "driving" inflation. The conjecture that generic expanding solutions of Einstein's field equations in vacuum with $\Lambda > 0$,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0, \tag{1.1}$$

approach the de-Sitter solution asymptotically is known as the cosmic no-hair conjecture [14, 16]. Recall that the de-Sitter solution is a homogeneous and isotropic inflationary solution of Eq. (1.1). If this conjecture were true, the asymptotic state of generic solutions would be characterized by homogenization, isotropization and exponential expansion.

Although there is some support for this conjecture in special situations [29, 30, 12, 7, 20, 17, 24], the general case remains unclear due to the complexity of Einstein's field equations. From this point of view, a particularly interesting family of solutions of Eq. (1.1) is the class of Nariai solutions which incorporates the (standard) Nariai solution [22, 23]. Their relevance for our questions results from the fact that they are *simple* solutions which are *not* consistent with the cosmic no-hair picture. If the cosmic no-hair conjecture is true, as is usually expected, this family of solutions must be non-generic in a certain sense and in particular unstable under arbitrary small perturbations. The perturbed solutions should generically either collapse and form a singularity in a given time direction, or if there is expansion, they should approach the de-Sitter solution. The main question for this and the second paper [4] is whether the expected instability of the Nariai solutions under perturbations is true, and if so, how this is realized dynamically.

Let us list some of the main assumptions. Throughout both papers, we will restrict our attention to cosmological spacetimes by which we mean 4-dimensional globally hyperbolic Lorentzian manifolds with compact spatial topology. It will turn out that spatial $\mathbb{S}^1 \times \mathbb{S}^2$-topology will be of particular interest for us. By cosmological solutions we mean cosmological spacetimes which solve Eq. (1.1) for $\Lambda > 0$. This means that all our results are purely classical, i.e. all quantum effects are ignored. In general, when we speak of a perturbation of a Nariai solution, we mean a cosmological solution of the fully non-linear Einstein's field equations Eq. (1.1) whose data, on some Cauchy surface, is close to the data on a Cauchy surface of a given Nariai solution. By "close" we mean that two data

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sets should deviate not too much with respect to some reasonable norm in the initial data space. In principle, we are interested in generic perturbations without symmetries. In practice, however, we must make simplifying assumptions, and this paper focuses on the spatially homogeneous case as a first step. In the second paper [4], we continue these investigations in a more complicated, in particular inhomogeneous class of perturbations.

This paper is organized as follows. In Section 2, we collect our conventions and notation for both papers. Section 3 contains a central part of this paper. In Section 3.1, we give a short discussion of the cosmic no-hair conjecture. We discuss conformal boundaries and their relevance for our interests here. Then, in Section 3.2, we introduce the (standard) Nariai solution and list its main properties. We analyze its asymptotics and define "Nariai asymptotics". We discuss its relation to the cosmic no-hair conjecture and prove that the Nariai solution has no smooth conformal boundaries. Afterwards, we introduce the family of generalized Nariai solutions in Section 3.3. The standard Nariai solution introduced earlier is a particular member of this family. We show that all generalized Nariai solutions are locally isometric to the (standard) Nariai solution, but in general have different global properties. In any case, all of them have Nariai asymptotics at least in one time direction. Although not all members in this family are relevant for our interests here by themselves, there is, first, an interesting property of their spatially homogeneous perturbations, as we find in Section 4, and, second, it will turn out that the standard Nariai solution is not sufficient for the particular approach taken in the second paper. In the following, when we speak of a "Nariai solution", we mean a generalized Nariai solution. The next central part of this paper is Section 4. We discuss spatially homogeneous solutions of Eq. (1.1) for spatial $\mathbb{S}^1 \times \mathbb{S}^2$-topology, also known as the Kantowski-Sachs family, close to (and including) the Nariai solutions. The cosmic no-hair conjecture has been studied before in this family in [30, 31, 7, 20]. We rederive those results which are of relevance for the subsequent work and make some new observations. The precise description of the instability in the spatially homogeneous case is of relevance for our second paper [4]. The paper is concluded with a summary and a short outlook to the second paper in Section 5.

2 Notation and conventions

Throughout this and the second paper [4], we assume Einstein's summation convention when we write tensorial expressions. Any tensor will either be represented by the abstract symbol, say, T or by abstract index notation, e.g. $T^{\mu}_{\ \nu}$, depending on the context. Note, however, that when we write such an indexed object, we can mean either the abstract tensor or a special component with respect to some basis. Hopefully, this will always be clear from the context. Our convention is that Greek indices $\mu, \nu, \ldots = 0, \ldots, 3$ refer to some choice of local spacetime coordinates, whereas Latin indices $i, j, \ldots = 0, \ldots 3$ represent indices with respect to some orthonormal frame field. When we have chosen a time coordinate $t = x^0$, then Greek indices $\alpha, \beta, \ldots = 1, 2, 3$ represent spatial coordinates, and for a choice of frame $\{e_i\}$ with timelike vector field e_0, Latin indices $a, b, \ldots = 1, 2, 3$ refer to spatial frame indices. Writing $\{e_i\}$ just means the collection of tangent vector

fields $\{e_0, \ldots, e_3\}$. If a 2-indexed object is written in brackets, for instance $(T^\mu_{\ \nu})$, we mean the matrix of its components, where the first index labels the lines of the matrix and the second one the columns.

3 Cosmic no-hair and the Nariai spacetimes

3.1 The cosmic no-hair conjecture

Conjecture 1 (Cosmic no-hair conjecture) Any generic future causal geodesically complete asymptotically future expanding cosmological solution of the vacuum Einstein's field equations with $\Lambda > 0$ is foliated by Cauchy surfaces which approach a homogeneous and isotropic foliation of the de-Sitter solution locally asymptotically to the future.

We say that a future causal geodesically complete cosmological solution "expands asymptotically to the future" provided there is a foliation of Cauchy surfaces labeled by time t whose mean curvature defined with respect to the future directed timelike unit normal is strictly positive for large t. When the solution is not necessarily future causal geodesically complete, but there is a foliation with strictly negative mean curvature on a relevant region of the maximal globally hyperbolic extension, we say that the solution "collapses". Note that these definitions rule out cosmological black hole spacetimes, for example the Schwarzschild-de-Sitter solution below, which are characterized by the existence of both collapsing and expanding regions. By a "homogeneous and isotropic foliation of the de-Sitter solution", we mean a family of Cauchy surfaces forming a foliation in the de-Sitter spacetime with a smooth transitive isometric action of a 6-dimensional Lie group on each leaf. The main example is given by the standard $t = const$-surfaces in the de-Sitter solution. In the following, we will often speak of a solution satisfying the "cosmic no-hair picture", when we mean that the solution has a foliation with the asymptotic properties of the conjecture.

It is difficult to find a precise formulation of this conjecture for several reasons. Some problems are listed in the introduction of [29]. We just note that the precise definitions of the notions of "genericity" and "approach" are not fixed by the conjecture. Apart from this, it is often difficult in practice to identify a foliation with the properties above for a given solution. In the spatially homogeneous case, there are geometrically preferred foliations and the analysis can simplify. Indeed, there are several results in the literature case which yield conditions in the spatially homogeneous so that *homogeneous* foliations approach a homogeneous and isotropic foliation of the de-Sitter solution. The first theorem in this direction was found in [29] and is valid for all Bianchi solutions of Einstein's field equations with $\Lambda > 0$ and matter under certain conditions. Other results in the spatially homogeneous case are [30, 31, 7, 20, 17]. Further fundamentally important results in this context are the non-linear stability of the de-Sitter solution [12, 19] and the stability theorem in [24], which both imply the cosmic no-hair conjecture close to certain reference solutions without symmetry assumptions.

Since in practice it can be difficult to check whether a given spacetime satisfies the cosmic no-hair picture, we introduce a particularly important, sometimes simpler but

less general, condition now. In this and the following paper [4], this condition shows up naturally and is used. One way of discussing asymptotics of spacetimes geometrically is to study the existence and properties of conformal boundaries. Let us consider the class of solutions of Einstein's field equations in vacuum with $\Lambda > 0$, which develop smooth future conformal boundaries $\mathcal{J}^+$ [13, 11, 2], for the following paragraph. The analogous notion for the past time direction exists. These solutions are also called future asymptotically de-Sitter for a reason to become clear in a moment. Smooth conformal boundaries correspond to the infinite timelike future or past in our setting and hence their properties characterize the timelike asymptotics of the spacetimes. The conformal formalism is particularly useful in the context of our two papers due to the following results.

Proposition 2 Let a future causal geodesically complete cosmological solution for $\Lambda > 0$ be given. Suppose it has a non-empty future conformal boundary $\mathcal{J}^+$. Then, in a neighborhood of $\mathcal{J}^+$, there exists a local foliation of spacelike surfaces which approaches a homogeneous and isotropic foliation of the de-Sitter solution asymptotically.

A particular example is the de-Sitter solution itself. By "approach" we mean pointwise convergence along each timeline at $\mathcal{J}^+$. For the proof, let (M, g) be spacetime under consideration and $(\tilde{M}, \tilde{g}, \Omega)$ its smooth conformal extension[1]. We identify M with the interior of $\tilde{M}$. The conformal factor Ω is a smooth function which is strictly positive on M, so that $\Omega = 0$ and $d\Omega \neq 0$ on $\mathcal{J}^+$. The identity $g = \Omega^{-2}\tilde{g}$ holds on M. Because $\mathcal{J}^+$ is spacelike for solutions of the vacuum field equations with $\Lambda > 0$ [13], we can foliate a neighborhood U in $\tilde{M}$ by spacelike level sets of Ω. Let the future directed timelike unit normal tangent vector field of these surfaces with respect to $\tilde{g}$ be $\tilde{N}$, and with respect to g be N. Note that $N = \Omega\tilde{N}$, and that both N and $\tilde{N}$ are smooth on U with $\tilde{N}$ non-vanishing. We choose a smooth non-vanishing (not necessarily orthonormal) frame $\{b_i\}$ with $b_0 = \tilde{N}$ and b_a ($a = 1, 2, 3$) orthogonal to $\tilde{N}$ on U. In particular, all b_a are hence tangent to the level sets and spacelike. The components of the 2nd fundamental form of the level sets with respect to g and $\tilde{g}$, respectively, for the frame $\{b_i\}$ are

$$\chi_{ij} := g(N, b_i)g(\nabla_N N, b_j) + g(\nabla_{b_i} N, b_j), \quad \tilde{\chi}_{ij} := \tilde{g}(\tilde{N}, b_i)\tilde{g}(\tilde{\nabla}_{\tilde{N}} \tilde{N}, b_j) + \tilde{g}(\tilde{\nabla}_{b_i} \tilde{N}, b_j).$$

Here, ∇ and $\tilde{\nabla}$ are the Levi-Civita covariant derivative operators of g and $\tilde{g}$ respectively. It is a standard fact that $\chi_{0i} = 0$, $\chi_{[ij]} = 0$, and the same for $\tilde{\chi}_{ij}$. Moreover, one can check that

$$\chi_{ij} = \Omega^{-1}\tilde{\chi}_{ij} - \Omega^{-2}\tilde{N}(\Omega)\tilde{h}_{ij},$$

where $\tilde{h}_{ij}$ are the components of the metric induced on the level sets by $\tilde{g}$. The following decomposition is standard

$$\chi_{ij} = Hh_{ij} + \sigma_{ij},$$

where the expansion or mean curvature H is three times the trace of χ_{ij} with respect to g and the shear tensor σ_{ij} is the traceless part. The same is done for $\tilde{\chi}_{ij}$ with respect

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[1]Note that we use the opposite notation to [4]. In our work here, all quantities related to the conformal metric $\tilde{g}$ are marked with a tilde, while all quantities related to the physical metric g have no mark.

to $\tilde{g}$ yielding quantities $\tilde{H}$ and $\tilde{\sigma}_{ij}$. It follows that

$$H = \Omega\tilde{H} - \tilde{N}(\Omega), \quad \sigma^i{}_j = \Omega\tilde{\sigma}^i{}_j.$$

In the latter expression, the index has been raised with g for the quantity $\sigma^i{}_j$ and for $\tilde{\sigma}^i{}_j$ with $\tilde{g}$. Now, since the conformal extension $(\tilde{M}, \tilde{g}, \Omega)$ and the frame $\{b_i\}$ is smooth on the conformal boundary, the quantities $\tilde{H}$, and $\tilde{\sigma}^i{}_j$ take finite values on $\mathcal{J}^+$. Moreover, Einstein's field equations in vacuum with $\Lambda > 0$ imply [11]

$$-\tilde{N}(\Omega)\Big|_{\mathcal{J}^+} = \sqrt{\frac{\Lambda}{3}}.$$

Hence, we see that in a neighborhood of $\mathcal{J}^+$ (given by $\Omega = 0$),

$$H = \sqrt{\frac{\Lambda}{3}} + O(\Omega), \quad \sigma^i{}_j = O(\Omega).$$

The value of H is hence consistent with the expansion of the de-Sitter solution and the shear vanishes asymptotically at $\mathcal{J}^+$; note that this is the case in particular when $\{b_i\}$ is substituted by a frame which is either orthonormal with respect to g or $\tilde{g}$. Now, concerning the curvature of the spatial slices, we find the following. Since in general the Ricci tensor is invariant under conformal rescalings of the metric with constant conformal factors, the Ricci tensor r of the spatial metric h equals the Ricci tensor $\tilde{r}$ of the spatial metric $\tilde{h}$. Furthermore, the frame components $\tilde{r}$ for an orthonormal frame of $\tilde{h}$ have finite values on $\mathcal{J}^+$ by virtue of the same argument as before. This shows that the components of r with respect to an orthonormal frame of h must be $O(\Omega^2)$ close to $\mathcal{J}^+$. Thus the physical curvature of the slices approaches zero. In the same way as [29], we interpret this as the local approach to the de-Sitter solution which completes the proof.

We remark that it is a priori not clear whether the surfaces in the proof of the previous proposition can be extended to surfaces which are both level sets of Ω and Cauchy surfaces globally. If $\mathcal{J}^+$ is compact, which is the most important case in the following, this extension, however, is possible.

For later convenience, another notion [24] is useful in this context. One says that future late time observers in a future causal geodesically complete cosmological spacetime are "completely oblivious to topology" if there is a Cauchy surface Σ such that for all future inextendible causal curves γ, the intersection of Σ with the causal past of γ is contained in a subset of Σ homeomorphic to a Euclidean 3-ball. In this situation, observers have no information about the manifold properties of Σ whatsoever because they can only see a "topologically trivial" 3-ball at most. Note that the assumption of future causal geodesic completeness rules out cosmological black hole solutions. Clearly, there is also a past dual of this notion.

Proposition 3 For any future causal geodesically complete cosmological spacetime (M, g) with a spacelike smooth compact future conformal boundary, late time observers are completely oblivious to topology.

In order to prove this proposition, recall that due to the assumptions, the future conformal boundary $\mathcal{J}^+$ is a Cauchy surface [2] of the conformal extension $(\tilde{M}, \tilde{g}, \Omega)$ of (M, g). In particular $\mathcal{J}^+$ is homeomorphic to all Cauchy surfaces of (M, g). This implies that any given future directed inextendible causal curve of (M, g) can be extended in $(\tilde{M}, \tilde{g})$ to hit $\mathcal{J}^+$. Moreover, there is a neighborhood of $\mathcal{J}^+$ in $(\tilde{M}, \tilde{g})$ with a Gaussian time function τ with respect to $\mathcal{J}^+$ such that the $\tau = 0$-hypersurface equals $\mathcal{J}^+$ and all $\tau = const$-surfaces with $\tau < 0$ are Cauchy surfaces of (M, g). Now if we choose a $\tau_0 < 0$ with $|\tau_0|$ small enough, then the $\tau = \tau_0$-surface Σ_0 has the required properties. That is, choose any future directed inextendible causal curve γ on (M, g) with future endpoint $p \in \mathcal{J}^+$. Choose a neighborhood V of p in $\mathcal{J}^+$ homeomorphic to a 3-ball. Let V_0 be the set in Σ_0 obtained from U via the past flow of the Gaussian time vector field. Then the causal past of p intersected with Σ_0 is contained in V_0 if $|\tau_0|$ is sufficiently small. Now the proposition follows because the causal past of p comprises the causal past of γ.

Since the de-Sitter solution possesses a smooth compact $\mathcal{J}^+$, Proposition 3 implies that late time observers are completely oblivious to topology. If a solution obeys the cosmic no-hair picture with respect to some foliation and hence approaches the de-Sitter solution in a "sufficiently well behaved" manner to the future, one can expect that it has the same property. Based on this argument, we conjecture the following alternative to the cosmic no-hair conjecture: For generic future causal geodesically complete future asymptotically expanding cosmological solutions of the vacuum field equations with $\Lambda > 0$, future late time observers are completely oblivious to topology.

3.2 Standard Nariai spacetime

3.2.1 Basic properties

Let Λ be an arbitrary number. Consider the Lorentzian manifold (M, g) given by

$$M = \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2), \quad g = \frac{1}{\Lambda} \left(-dt^2 + \cosh^2 t \, d\rho^2 + g_{\mathbb{S}^2} \right), \tag{3.1}$$

with the standard coordinate ρ on the manifold $\mathbb{S}^1$, the standard round unit metric $g_{\mathbb{S}^2}$ on $\mathbb{S}^2$ and the time coordinate $t \in \mathbb{R}$. This spacetime is called the Nariai spacetime. It was first discussed by Nariai [22, 23] and later reconsidered in various works; an overview of references is given by [6]. It is also sometimes called Weber solution due to the work in [30, 31]. This solution has turned out to be very interesting from the semi-classical point of view because it has certain extremal horizon properties [6]. We will not discuss such issues here. Since we also want to consider "generalized" Nariai spacetimes in Section 3.3, we often speak of the "standard" Nariai spacetime where necessary. We will also often consider the universal cover of the standard Nariai spacetime. This is the simply connected Lorentzian manifold $(\tilde{M}, g)$ with $\tilde{M} = \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2)$ and the metric g above.

The standard Nariai spacetime has the following properties.

(i) It is an analytic cosmological solution of Einstein's field equations in vacuum with a cosmological constant $\Lambda > 0$. Hence we will often also speak of the (standard)

Nariai *solution*. The $t = const$-hypersurfaces are spacelike Cauchy surfaces of topology $\mathbb{S}^1 \times \mathbb{S}^2$.

(ii) It is geodesically complete because it is the direct product of the 2-dimensional de-Sitter spacetime and a rescaled round 2-sphere.

(iii) The representation Eq. (3.1) of the universal cover of the standard Nariai spacetime is the maximal analytic extension. Maximality follows from geodesic completeness. Uniqueness is implied by the version of Theorem 6.3 in [18] for Lorentzian metrics, or by Theorem 4.6 from [8] for the special case of geodesic completeness. In the literature one finds other representations of the Nariai metric, see the references above, which are not always geodesically complete and hence not necessarily maximal.

(iv) It is spatially homogeneous with Kantowski-Sachs symmetry group $G = \mathbb{R} \times SO(3)$.

(v) The Nariai solution (M, g) has "Nariai asymptotics"; a term which we explain now.

3.2.2 Nariai asymptotics and cosmic no-hair

A particularly peculiar property of the Nariai solution is its time dependence. While the $\mathbb{S}^1$-factor of the spatial slices expands exponentially for increasing positive t, the volume of the $\mathbb{S}^2$-factor stays constant. Thus, the expansion of this solution is anisotropic in the sense that the shear tensor of the $t = const$-slices never approaches zero; cf. the proof of Proposition 2. This implies that the cosmic no-hair picture does not hold with respect to the foliation of $t = const$-surfaces. So we must ask whether there exist other foliations of the Nariai solution, presumably based on non-symmetric surfaces, for which the cosmic no-hair picture is attained. In this paper, we are not able to answer this question completely, and we stress that to our knowledge there is no result in the literature which implies that the Nariai solution contradicts the cosmic no-hair picture for *all* foliations. Nevertheless, we discuss a property of this solution now which suggests that the cosmic no-hair picture does not hold for the Nariai solution. Then we prove a weaker statement below, to the first time rigorously in the literature, that the Nariai solution does not possess smooth conformal boundaries.

Ringström [24] introduces the terminology that in a future causal geodesically complete spacetime, future late time observers are "oblivious to topology" if there is a Cauchy surface Σ such that there are no future directed inextendible causal curves whose causal pasts contain Σ. The reader should distinguish this from the notion "completely oblivious to topology" introduced in Section 3.1. Again, there clearly exists a corresponding past dual of this notion, but we restrict to the future. In physical terms this means that all light rays are confined to a subregion of the spatial slices due to the rapid expansion.

It is evident that for spacetimes whose observers are completely oblivious to topology, observers are also oblivious to topology. In particular, for the de-Sitter solution it thus follows that observers are oblivious to topology, and indeed also for the Nariai solution. However, the Nariai solution is an example with the following property.

Proposition 4 For the Nariai solution, and also for its universal cover, future late time observers are oblivious to topology, but not completely oblivious to topology.

In the following, we say that all solutions with this property have "Nariai asymptotics". This proposition is a consequence of an even stronger result by Ringström [24]. That is, regardless of the choice of Cauchy surface Σ and future directed inextendible causal curve γ in the Nariai solution, the intersection of Σ and the causal past of γ is not contained in a set homeomorphic to a 3-ball. In simple words, although observers in the Nariai solutions do not have full information about the spatial topology due to the rapid expansion, they will always have *some* information because the anisotropy of the expansion allows to see the full $\mathbb{S}^2$-factor asymptotically. Ringström's original proof of this statement in Lemma 21 in [24] uses the compactness of the Cauchy surfaces of the standard Nariai solution and hence does not apply to the universal cover. However, his argument can be rescued for the universal cover as follows. Let Σ be a Cauchy surface and γ a future directed inextendible causal curve in the universal cover spacetime. Now the main new observation is that the closure of $J^-(\gamma) \cap \Sigma$ is compact for the Nariai metric. Hence, when we assume that this set is contained in $B \subset \Sigma$ homeomorphic to a 3-ball, we can assume in particular that $\bar{B}$ is compact. Ringström's contradiction argument at the end of his original proof can now be applied with respect to a value of τ chosen large enough so that the $t = \tau$-surface S_τ is strictly to the future of $\bar{B}$.

A particular consequence of Ringström's result above is that the Nariai solution does not possess a smooth conformal boundary. This has been conjectured in [14] and in many references afterwards. The conjecture was based on the simplest conformal rescaling of the metric Eq. (3.1) by means of a conformal factor only depending on t because this does not lead to a smooth conformal compactification. However, in order to turn this statement into a theorem, one needs to show that there exists no conformal rescaling at all which would yield a smooth conformal compactification.

Proposition 5 The standard Nariai solution does not have even a patch of a smooth conformal boundary. The same is true for its universal cover.

Suppose the contrary is the case, and an open patch of a smooth future conformal boundary $\mathcal{J}^+$, not necessarily compact, exists for the Nariai solution (M, g). Since (M, g) is a cosmological solution of Eq. (1.1), $\mathcal{J}^+$ is spacelike [13]. Then there exists a future directed inextendible causal curve γ of (M, g) which can be extended in the conformally extended spacetime $(\tilde{M}, \tilde{g})$ so that it hits $\mathcal{J}^+$ in the point p. Let V be a neighborhood of p in $\mathcal{J}^+$ homeomorphic to a 3-ball; this neighborhood exists thanks to our assumption that $\mathcal{J}^+$ consists at least of an open patch. There is a neighborhood U of V in $(\tilde{M}, \tilde{g})$ with a Gaussian time function τ such that the $\tau = 0$-hypersurface equals V and all $\tau = const$-surfaces with $\tau < 0$ are spacelike surfaces of (M, g). Now, if we choose $\tau_0 < 0$ with $|\tau_0|$ small enough, then any future directed inextendible causal curve with p as its future endpoint must intersect the $\tau = \tau_0$-hypersurface. Since this surface is homeomorphic to a 3-ball, because V has this property, and since it can be extended to a Cauchy surface, because it is spacelike, we have found a contradiction to Ringström's statement above, that no Cauchy surface and causal curve with this property exist in

the Nariai solution. Hence, our claim is proven.

According to our speculation about cosmic no-hair at the end of Section 3.1, all this suggests that the Nariai solution does not satisfy the cosmic no-hair picture and hence motivates our claim that it should be non-generic among solutions of the field equations.

3.3 Generalized Nariai spacetimes

3.3.1 Definition and general properties

Consider the Lorentzian manifold (M, g) with

$$g = \frac{1}{\Lambda}\left(-dt^2 + \Phi(t)^2\, d\rho^2 + g_{\mathbb{S}^2}\right) \tag{3.2a}$$

with

$$\Phi(t) = \Phi_0 \cosh t + \Phi_0' \sinh t \tag{3.2b}$$

for arbitrary $\Lambda > 0$, $\Phi_0 > 0$ and $\Phi_0' \in \mathbb{R}$ and with

$$M = I \times (\mathbb{S}^1 \times \mathbb{S}^2).$$

Here, I is the unique maximal connected open interval in $\mathbb{R}$ with $0 \in I$ and $\Phi|_I > 0$. Note that Φ can change its sign at most once. We will henceforth refer to this family as the "generalized Nariai spacetimes". Clearly, for $\Phi_0 = 1$ and $\Phi_0' = 0$, we recover the standard Nariai solution. All generalized Nariai spacetimes are analytic solutions of the vacuum Einstein's field equations with a cosmological constant Λ. They are globally hyperbolic, but not necessarily causal geodesically complete; as we discuss further now. They are spatially homogeneous of Kantowski-Sachs type. We will also often consider the universal cover. This is the simply connected Lorentzian manifold $(\tilde{M}, g)$ with $\tilde{M} = \mathbb{R} \times (\mathbb{R} \times \mathbb{S}^2)$ and the metric g above. Note that our family of generalized Nariai solution is related to, but distinct from the family with the same name defined in [9] in Eq. (1.2). In our language, the latter family assumes $\Phi_0' = 0$ but arbitrary signs of Λ, while our family restricts to $\Lambda > 0$ with arbitrary Φ_0'.

3.3.2 Global behavior

Let us define

$$\sigma_0 := \Phi_0^2 - (\Phi_0')^2. \tag{3.3}$$

One finds easily that Φ defined in Eq. (3.2) has a real zero if and only if $\sigma_0 < 0$. There cannot be more than one real zero. In particular, $I = \mathbb{R}$ if and only if $\sigma_0 \geq 0$. Similarly, one shows that Φ has one and only one local minimum if and only if $\sigma_0 > 0$.

Let us consider the universal cover of a generalized Nariai solution in order to avoid problems with identifications on the $\mathbb{S}^1$-factor in the following arguments. Furthermore, let us fix once and for all a value of Λ. The discussion above implies that if $\sigma_0 > 0$, we can shift the time coordinate t so that $t = 0$ is the time of the minimum of Φ and hence $\Phi_0' = 0$ and $\Phi_0 > 0$. Note that σ_0 is invariant under time translation. Then,

by rescaling the ρ-coordinate we can choose $\Phi_0 = 1$. Hence the universal cover of any generalized Nariai solutions with $\sigma_0 > 0$ is isometric to the universal cover of the standard Nariai solution with $\sigma_0 = 1$. The isometry involved here is analytic. For $\sigma_0 = 0$, the function Φ has neither a zero nor a local extremum, and $\Phi_0' = \pm\Phi_0$. By a change of time direction and rescaling of the ρ-coordinate, we can always arrange that $\Phi_0' = \Phi_0 = 1$. Hence, the universal cover of any generalized Nariai solution with $\sigma_0 = 0$ is isometric to the universal cover of the generalized Nariai solution with $\Phi_0' = \Phi_0 = 1$. The isometry is again analytic. Finally, consider the universal cover of a generalized Nariai solution with $\sigma_0 < 0$. By a rescaling of the ρ-coordinate, we can choose $\sigma_0 = -1$. Now, since $\Phi : I \to \mathbb{R}^{>0}$ is surjective, there is a time translation which gives for instance $\Phi_0 = (e^2 - 1)/2e$. Since $\sigma_0 = -1$, we can transform $t \mapsto -t$ if necessary in order to get $\Phi_0' = (e^2 + 1)/2e$. Thus the universal cover of any generalized Nariai solutions with $\sigma_0 < 0$ is isometric to the universal cover of the generalized Nariai solution given by $\Phi_0 = (e^2 - 1)/2e$, $\Phi_0' = (e^2 + 1)/2e$, and the isometry is analytic.

Hence, we have identified three analytic isometry classes of generalized Nariai solutions given by the sign of the constant σ_0. Are they distinct, or related for instance to the standard Nariai solution in some way? For the case $\sigma_0 > 0$ we have already shown that the universal cover of the corresponding generalized Nariai solution is globally isometric to the universal cover of the standard Nariai solution. Let us consider the case $\sigma_0 < 0$. Without loss of generality we can assume that $\Phi_0 = (e^2 - 1)/2e$, $\Phi_0' = (e^2 + 1)/2e$ and hence that $I = (-1, \infty)$. Let us denote the universal cover of this generalized Nariai solution by $(\hat{M}, \hat{g})$ and its coordinates by $(\hat{t}, \hat{\rho}, \hat{\theta}, \hat{\phi})$. The universal cover of the standard Nariai solution is referred to as (M, g) with coordinates (t, ρ, θ, ϕ). Consider the map

$$\Psi_- : \hat{M} \to M, \quad (\hat{t}, \hat{\rho}, p) \mapsto (t(\hat{t}, \hat{\rho}), \rho(\hat{t}, \hat{\rho}), p)$$

where p denotes an arbitrary point on $\mathbb{S}^2$ and where

$$t(\hat{t}, \hat{\rho}) = \operatorname{arcsinh}\left(\sinh(\hat{t} + 1)\sqrt{\sinh^2\hat{\rho} + 1}\right),$$

$$\rho(\hat{t}, \hat{\rho}) = \arcsin\left(\frac{\sinh(\hat{t} + 1)\sinh\hat{\rho}}{\sqrt{\left(\sinh^2\hat{\rho} + 1\right)\sinh^2(\hat{t} + 1) + 1}}\right) + \pi.$$

One can check that the map Ψ_- is an analytic embedding and, considered as a map onto its image, an isometry. It is obvious that the same statement would be true if we added other constants to $\rho(\hat{t}, \hat{\rho})$. Before we discuss this map further, let us continue with the case $\sigma_0 = 0$. Let us use the same notation as before in this case for the map
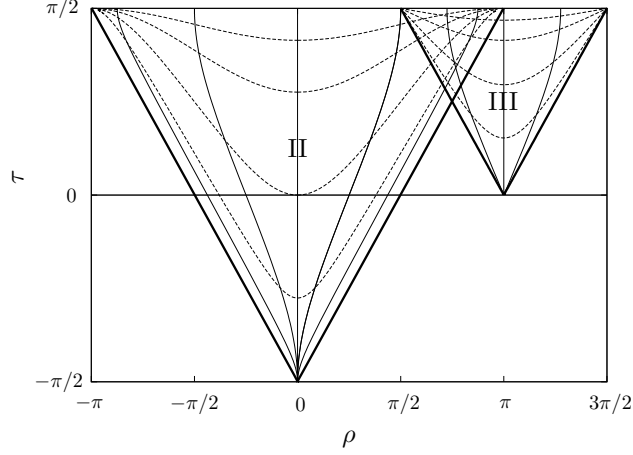
$$\Psi_0 : \hat{M} \to M.$$

Fig. 1: Embeddings of generalized Nariai solutions into the standard one

When we set

$$t(\hat{t}, \hat{\rho}) = \operatorname{arcsinh}\left(\frac{e^{\hat{t}}\left(\hat{\rho}^2 + 1\right) - e^{-\hat{t}}}{2}\right)$$

$$\rho(\hat{t}, \hat{\rho}) = \begin{cases} -\arccos\left(\dfrac{e^{-\hat{t}} - e^{\hat{t}}\left(\hat{\rho}^2 + 1\right) + 2e^{\hat{t}}}{\sqrt{\left(e^{\hat{t}}(\hat{\rho}^2 + 1) - e^{-\hat{t}}\right)^2 + 4}}\right) & \hat{\rho} < 0, \\[20pt] 0 & \hat{\rho} = 0 \;, \\[20pt] \arccos\left(\dfrac{e^{-\hat{t}} - e^{\hat{t}}\left(\hat{\rho}^2 + 1\right) + 2e^{\hat{t}}}{\sqrt{\left(e^{\hat{t}}(\hat{\rho}^2 + 1) - e^{-\hat{t}}\right)^2 + 4}}\right) & \hat{\rho} > 0, \end{cases}$$

we find that Ψ_0 is also an analytic isometric embedding.

Hence we have shown for the universal covers, that all generalized Nariai solutions can be embedded isometrically into the standard Nariai solution. We visualize this in Fig. 1. In this figure, the horizontal axis represents the coordinate ρ and the vertical axis shows

$$\tau := 2\arctan(\tanh(t/2)).$$

We consider again the universal cover of the standard Nariai solution. This compactified time coordinate has the property that $t \to \pm\infty$ corresponds to $\tau \to \pm\pi/2$. Each point in this diagram determines a 2-sphere in the standard Nariai solution. In these coordinates, null curves which are constant on the $\mathbb{S}^2$-factor are straight lines and their angles have the Minkowski value. Hence the picture appears like a Penrose diagram. However, note that this would not be the case if we considered general null curves. For the case $\sigma_0 > 0$, the corresponding generalized Nariai solution is isometric globally to the standard Nariai solution. For $\sigma_0 = 0$, the region covered by the embedding Ψ_0 is given by the region II in Fig. 1 for $\Phi_0' > 0$ which we can always assume. The thin dashed curves represent $\hat{t} = const$-surfaces and the non-dashed thin curves $\hat{\rho} = const$-surfaces. The black bold

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lines correspond to the limit $\hat{t} \to -\infty$. For the case $\sigma_0 < 0$ and $\Phi_0' > 0$, the region III is covered by the map Ψ_-. Here the black bold lines correspond to the curves $\hat{t} \to -1$ for our standard choice of parameters.

The above implies that the maximal analytic extensions of the universal covers of all generalized Nariai solutions are isometric to the universal cover of the standard Nariai solution. Our discussion furthermore proves that for $\sigma_0 \leq 0$, all generalized Nariai solutions are causal geodesically incomplete. Finally, after a change of time direction if necessary, all generalized Nariai solutions have Nariai asymptotics to the future.

All these statements hold for the universal covers and yield globally defined isometric embeddings. Returning to the original manifolds, we see that the maps corresponding to Ψ_0 and Ψ_- are not smooth globally for $\sigma_0 \leq 0$, however are still local isometries.

4 Instability of Nariai asymptotics within the Kantowski-Sachs family

4.1 Ansatz and general solution

In the following, we explain the non-genericity of the Nariai solutions within the spatially homogeneous class of solutions. We start by first writing the general solution of the field equations which comprises the generalized Nariai solutions. Then we define spatially homogeneous perturbations and study their properties.

More precisely, we consider the class of spacetimes with a manifold given by $M = \mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$ and a metric g which is invariant under a global smooth effective action of the Kantowski-Sachs group, whose orbits are spacelike Cauchy surfaces. This allows us to write the metric in the form [28]

$$g = \frac{1}{\Lambda} \left(-dt^2 + \bar{F}(t)d\rho^2 + \bar{G}(t)g_{\mathbb{S}^2} \right) \tag{4.1}$$

with similar conventions as before. The functions $\bar{F}$ and $\bar{G}$ play a similar role as the single scale factor a in Friedmann-Robertson-Walker solutions [28]. We factor out $1/\Lambda$ so that all quantities in the bracket are dimensionless. Recall that Λ has the dimension Length^{-2} in geometric units and that the coordinate components of g have dimension Length^2 for dimensionless coordinates. In order to simplify the problem of dimensions, we will always assume $\Lambda = 1$ in the following. A metric g, which is a solution of Eq. (1.1) with $\Lambda = 1$, yields a metric $\hat{g} = g/\hat{\Lambda}$ which solves Eq. (1.1) with $\Lambda = \hat{\Lambda} > 0$. This means that the choice $\Lambda = 1$ means no loss of generality, and hence allows us to restrict to dimensionless quantities in the following.

The metric given by Eq. (4.1) is well-defined as long as $\bar{F}$ and $\bar{G}$ are positive. However, in this representation, it is difficult to discuss the limit when $\bar{F}$ or $\bar{G}$ become zero. This is the reason why we prefer to work with "Eddington-Finkelstein"-like coordinates (s, μ, θ, ϕ) on $\mathbb{R} \times (\mathbb{S}^1 \times \mathbb{S}^2)$, related to the above Gaussian coordinates by

$$\sqrt{\bar{F}(t)}\, dt = ds, \quad d\rho = 1/\bar{F}(t)\, ds + d\mu,$$

as long as $\bar{F}$ is positive. In these coordinate the metric takes the form

$$g = 2ds d\mu + F(s)d\mu^2 + G(s)g_{\mathbb{S}^2}$$

where

$$F(s) = \bar{F}(t(s)), \quad G(s) = \bar{G}(t(s)).$$

Note that s is a null coordinate. The symmetry hypersurfaces are the $t = const$-, or equivalently the $s = const$-surfaces. As before for the coordinate ρ, the function μ is the coordinate on the $\mathbb{S}^1$-factor, generated by the translation subgroup of the Kantowski-Sachs symmetry group. The motivation for this choice of coordinates is that the metric is well-behaved when F changes sign, the curvature is bounded as long as F is smooth and $G > 0$, and hence the metric can sometimes be extended even where the symmetry hypersurfaces change their causal character.

Let us assume in the following that the initial hypersurface for the initial value problem of Eq. (1.1) under these assumptions is given by $s = 0$ and is spacelike, i.e. $F(0), G(0) > 0$. It is clear that an s-neighborhood of the initial hypersurface is globally hyperbolic and that the $s = const$-hypersurfaces are spacelike compact Cauchy surfaces. One easily computes the mean curvature of the $s = const$-hypersurfaces

$$H(s) = \frac{1}{6} \left(\frac{1}{\sqrt{F(s)}} \frac{dF}{ds} + 2 \frac{\sqrt{F(s)}}{G(s)} \frac{dG}{ds} \right), \tag{4.2}$$

which holds as long as the surface is spacelike.

In the coordinates (s, μ, θ, ϕ), Einstein's field equations in vacuum with $\Lambda = 1$ reduce to

$$0 = \ddot{G}(s) - \frac{\dot{G}^2(s)}{2G(s)} \tag{4.3a}$$

$$0 = \ddot{F}(s) + \frac{\dot{F}(s)\dot{G}(s)}{G(s)} - 2 \tag{4.3b}$$

$$0 = 1 - \frac{1}{G(s)} - \frac{\dot{F}(s)\dot{G}(s)}{2G(s)} - \frac{F(s)\dot{G}^2(s)}{4G^2(s)}, \tag{4.3c}$$

where a dot represents derivatives with respect to s. The initial value problem for these equations is well-posed under our assumptions, and for given data on the $s = 0$-hypersurface satisfying the constraint Eq. (4.3c), it has a unique solution maximally extended in the coordinates (s, μ, θ, ϕ). Indeed, all solutions can be written explicitly as

$$F(s) = F_* + \dot{F}_* s \frac{2}{H_*^{(0)}s + 2} + s^2 \frac{H_*^{(0)}s + 6}{3H_*^{(0)}s + 6},$$

$$G(s) = \frac{G_*}{4}(H_*^{(0)}s + 2)^2,$$

with constants $G_* > 0$, $F_* > 0$, $\dot{F}_*$ and $H_*^{(0)}$ corresponding to the data of F and G on the initial hypersurface by

$$F_* = F(0), \quad G_* = G(0), \quad \dot{F}_* = (\partial_s F)(0), \quad H_*^{(0)} = (\partial_s G)(0)/G(0).$$

4.2 Characterization of the solutions

Consider arbitrary data consistent with the constraint for $H_*^{(0)} = 0$. The corresponding solution – denoted (M, g) in the following – is

$$F(s) = F_* + \dot{F}_* s + s^2, \quad G(s) = 1,$$

with $F_* > 0$ and $\dot{F}_*$ arbitrary. We find that (M, g) is an analytic extension of the generalized Nariai solution – denoted by $(\tilde{M}, \tilde{g})$ for now – given by

$$\Phi_0 = \sqrt{F_*}, \quad \Phi_0' = \dot{F}_*/2. \tag{4.4}$$

The corresponding analytic isometric embedding $\tilde{M} \to M$ is given by

$$s(t) = (\partial_t \Phi)(t) - \Phi_0',$$

$$\mu(\rho, t) = \rho - \begin{cases} \frac{1}{\sqrt{\sigma_0}} \left(\arctan \frac{(\partial_t \Phi)(t)}{\sqrt{\sigma_0}} - \arctan \frac{\Phi_0'}{\sqrt{\sigma_0}} \right) & \sigma_0 > 0, \\ -\frac{1}{(\partial_t \Phi)(t)} + \frac{1}{\Phi_0'} & \sigma_0 = 0, \\ -\frac{1}{\sqrt{|\sigma_0|}} \left(\operatorname{arctanh} \frac{(\partial_t \Phi)(t)}{\sqrt{\sigma_0}} - \operatorname{arctanh} \frac{\Phi_0'}{\sqrt{\sigma_0}} \right) & \sigma_0 < 0. \end{cases}$$

This is why we call data for the Kantowski-Sachs initial value problem with $H_*^{(0)} = 0$ Nariai data. With this we find the following further aspects for generalized Nariai solutions. In these discussions we identify $(\tilde{M}, \tilde{g})$ with the image of the isometric embedding in (M, g).

(i) If $\sigma_0 > 0$ for $(\tilde{M}, \tilde{g})$, then (M, g) and $(\tilde{M}, \tilde{g})$ coincide, and both correspond to the maximal globally hyperbolic development of Nariai data. This follows from geodesic completeness.

(ii) If $\sigma_0 < 0$ for $(\tilde{M}, \tilde{g})$, then $(\tilde{M}, \tilde{g})$ is only defined for $t \in I \neq \mathbb{R}$, and (M, g) is an extension of $(\tilde{M}, \tilde{g})$. From Eq. (4.2), we see that the mean curvature of the $s = const$-surfaces, which are Cauchy surfaces of $(\tilde{M}, \tilde{g})$, blows up uniformly exactly at the boundary of the embedding. This implies that the solution cannot be extended as a globally hyperbolic spacetime. We conclude that $(\tilde{M}, \tilde{g})$ is the maximal globally hyperbolic development of Nariai data. However, since the curvature is bounded, it is extendible and the boundary of $(\tilde{M}, \tilde{g})$ in (M, g) is a Cauchy horizon of topology $\mathbb{S}^1 \times \mathbb{S}^2$, generated by closed null curves.

(iii) If $\sigma_0 = 0$ for $(\tilde{M}, \tilde{g})$, then $(\tilde{M}, \tilde{g})$ is defined for all $t \in \mathbb{R}$. After a change of time direction, if necessary, we have that

$$\lim_{t \to -\infty} s(t) = -\Phi_0', \quad \lim_{t \to +\infty} s(t) = +\infty.$$

Hence, (M, g) is an extension of $(\tilde{M}, \tilde{g})$. In this case, the mean curvature of the $s = const$-hypersurfaces stays bounded in the limit $t \to -\infty$. Based on these
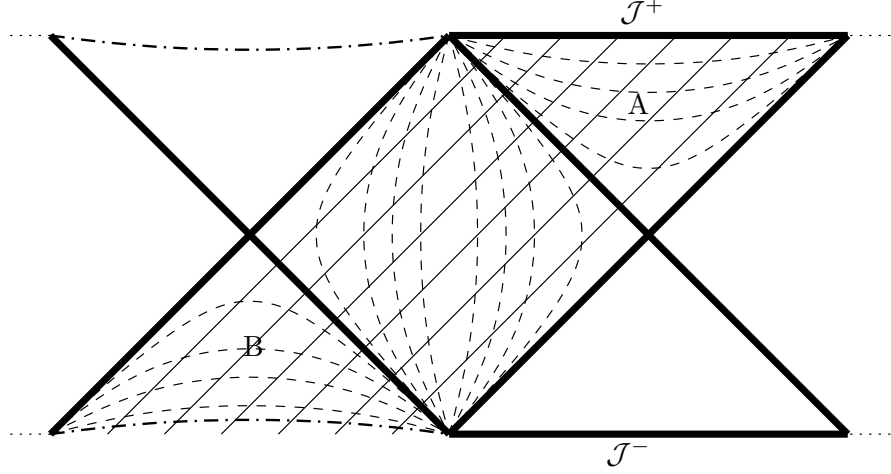
Fig. 2: Penrose diagram of Schwarzschild-de-Sitter in the mass regime $0 < S < 1/3$ and $\Lambda = 1$

simple arguments it is not possible to conclude that $(\tilde{M}, \tilde{g})$ is the maximal globally hyperbolic development of Nariai data with $\sigma_0 = 0$, because we cannot exclude the possibility that there exist other extensions which are globally hyperbolic. We will not discuss this issue further here, since, as far as we can see at the moment, the case $\sigma_0 = 0$ is a borderline case of no particular interest for us.

Now let us consider the "generic" case $H_*^{(0)} \neq 0$. Then we can introduce the following new coordinates

$$\hat{s} = \frac{\sqrt{G_*}}{2}(H_*^{(0)} s + 2), \quad \hat{\mu} = \frac{2}{H_*^{(0)}\sqrt{G_*}}\mu. \tag{4.5}$$

In these coordinates, the metric becomes

$$g = 2d\hat{s}d\hat{\mu} + \underbrace{\frac{(H_*^{(0)})^2 G_*}{4}F\left((2\hat{s}/\sqrt{G_*} - 2)/H_*^{(0)}\right)}_{=:\hat{F}(\hat{s})} d\hat{\mu}^2 + \hat{s}^2 g_{\mathbb{S}^2}.$$

After straightforward computations, substituting F_* by means of the constraint, we find

$$\hat{F}(\hat{s}) = -1 + \frac{2S}{\hat{s}} + \frac{1}{3}\hat{s}^2 \tag{4.6a}$$

with

$$S := G_*^{3/2}\left(\frac{1}{3} - \frac{1}{4}\dot{F}_* H_*^{(0)}\right). \tag{4.6b}$$

Hence we have shown that the Kantowski-Sachs solutions with $H_*^{(0)} \neq 0$ yield the Schwarzschild-de-Sitter solution with mass S in Eddington-Finkelstein coordinates.

Let us make a few comments about the Schwarzschild-de-Sitter solution by means of Fig. 2. More details are presented in [3]. The figure shows the Penrose diagram of

the Schwarzschild-de-Sitter solution with mass $0 < S < 1/3$ and $\Lambda = 1$ as an example. Bold diagonal lines denote the event and cosmological horizons, bold dashed curves refer to the singularity, and bold horizontal lines mark parts of the conformal boundary $\mathcal{J}^+$ and $\mathcal{J}^-$. Furthermore, thin dashed curves represent $s = const$-hypersurfaces and thin non-dashed curves $\mu = const$-hypersurfaces. The $s = const$-surfaces are spacelike in regions A and B, but timelike between the cosmological and black hole event horizons. The cosmological horizon and the event horizon become Cauchy horizons for the Cauchy development of any $s = const$-hypersurface in the regions A and B.

We remark that all the results here are consistent with the Birkhoff theorem for $\Lambda > 0$ proved in [27].

4.3 Perturbations of Nariai within the Kantowski-Sachs family

So far we have shown that all Kantowski-Sachs solutions of our initial value problem with $H_*^{(0)} \neq 0$ are locally isometric to subsets of Schwarzschild-de-Sitter solutions with mass S given by Eq. (4.6), while for $H_*^{(0)} = 0$, we get the generalized Nariai solutions. In order to simplify the discussion again, we will often go to the universal covers in the following without further comment. Let us fix a generalized Nariai solution (M, g) either by means of the parameters $\Phi_0 > 0$, $\Phi_0' \in \mathbb{R}$ or by $F_* > 0$, $\dot{F}_* \in \mathbb{R}$, $H_*^{(0)} = 0$ related by Eq. (4.4). Now, leave all these parameters fixed except for $H_*^{(0)}$, which we give some small but non-vanishing value. We denote the corresponding Kantowski-Sachs solution by $(\tilde{M}, \tilde{g})$ and call it the perturbation of (M, g).

For sufficiently small $|H_*^{(0)}| > 0$, we find that $(\tilde{M}, \tilde{g})$ is a Schwarzschild-de-Sitter solution with mass

$$S = \frac{1}{3} + \frac{\sigma_0}{8}(H_*^{(0)})^2 - \Phi_0' \frac{2(\Phi_0')^2 - 3\sigma_0}{24}(H_*^{(0)})^3 + O((H_*^{(0)})^4).$$

Note that these and some of the following formulas have been derived before, cf. [15, 10], for the standard Nariai solution. For sufficiently small $|H_*^{(0)}| > 0$, we have the following conclusions:

(i) Suppose a generalized Nariai solution (M, g) with $\sigma_0 > 0$ is given. Then the perturbations $(\tilde{M}, \tilde{g})$ are locally isometric to a Schwarzschild-de-Sitter solution with mass $S > 1/3$.

(ii) If $\sigma_0 < 0$, then $(\tilde{M}, \tilde{g})$ is locally isometric a Schwarzschild-de-Sitter solution with $0 < S < 1/3$.

(iii) If $\sigma_0 = 0$, then $(\tilde{M}, \tilde{g})$ has the following properties:

    (a) If $\Phi_0' H_*^{(0)} > 0$: locally isometric to a Schwarzschild-de-Sitter solution with $0 < S < 1/3$.

    (b) If $\Phi_0' H_*^{(0)} < 0$: locally isometric to a Schwarzschild-de-Sitter solution with $S > 1/3$.

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(c) The case $\Phi_0' H_*^{(0)} = 0$ can be excluded.

It is an interesting result that the three isometry classes of generalized Nariai solutions yield quite different perturbations despite of the fact that they are locally isometric.

For $(\tilde{M}, \tilde{g})$ for given finite s and μ and sufficiently small $|H_*^{(0)}| > 0$, we find

$$\hat{s} = 1 + \frac{\Phi_0' + s}{2} H_*^{(0)} + O((H_*^{(0)})^2),$$

$$\hat{\mu} = \mu \left(\frac{2}{H_*^{(0)}} - \Phi_0' - \frac{2(\Phi_0') + \sigma_0}{4} H_*^{(0)} + O((H_*^{(0)})^2) \right).$$

Note that this suggests that generalized Nariai solutions can be considered as singular limits of Schwarzschild-de-Sitter solutions with masses as discussed above. That is, in the limit $H_*^{(0)} \to 0$, we have

$$M \to 1/3, \quad \hat{s} \to 1,$$

and further, for $H_*^{(0)} \searrow 0$, one gets

$$\hat{\mu} \to \begin{cases} -\infty & \mu < 0 \\ 0 & \mu = 0 \\ \infty & \mu > 0. \end{cases}$$

For $H_*^{(0)} \nearrow 0$, the signs for the limits of $\hat{\mu}$ turn around. We will not consider the consequences of this, but see for instance [15, 14]. However, note that μ and s have to be finite in order to give sense to these expansions, and thus they do not yield information about the asymptotics of the generalized Nariai solution. This and the fact that the limit of $\hat{\mu}$ is singular in this "Nariai limit of the Schwarzschild-de-Sitter class" is obvious from the formulas above, but often not pointed out clearly in the literature. For instance, the author of [6] is led to the statement that the Nariai solution is "the largest possible black hole in de Sitter space", i.e. a Schwarzschild-de-Sitter solution with mass $M = 1/3$, by the analogous considerations in [15]. At least from our point of view, this statement is misleading.

In order to compare the global properties of the perturbation $(\tilde{M}, \tilde{g})$ with those of the unperturbed Nariai solution (M, g), let us restrict to the example case $\sigma_0 < 0$. This is not the standard case in the literature, but will play a particular role in the following paper [4]. Let us consider Fig. 2 for the Penrose diagram of the Schwarzschild-de-Sitter solution with $0 < S < 1/3$. We can change the time direction so that $\Phi_0' > 0$ (note that $\Phi_0' = 0$ is not allowed in this class) for the unperturbed Nariai solution (M, g). Now let $H_*^{(0)}$ be sufficiently small and positive. In this case, the initial hypersurface of our Kantowski-Sachs initial value problem given by $s = 0$ corresponds to $\hat{s} = \hat{s}_0 > 1$. From the results derived above and the well-known root structure of $\hat{F}$, it follows that the initial hypersurface must correspond to a $\hat{s} = const$-Cauchy surface of region A in Fig. 2. Hence, we can say the following in this case:

- In the future time direction, while the unperturbed Nariai solution (M, g) has Nariai asymptotics, the perturbation $(\tilde{M}, \tilde{g})$ has a smooth future conformal boundary $\mathcal{J}^+$. In other words, while for the original solution the volume of the $\mathbb{S}^2$-factor is constant in time, an arbitrarily small initial positive expansion of the $\mathbb{S}^2$-factor determined by a small $H_*^{(0)} > 0$ leads to accelerated expansion of the $\mathbb{S}^2$-factor. Together with the accelerated expansion of the $\mathbb{S}^1$-factor present also in the unperturbed solution, this yields the existence of a smooth future conformal boundary $\mathcal{J}^+$.

- In the past time direction, both (M, g) and the perturbation $(\tilde{M}, \tilde{g})$ develop a Cauchy horizon, which in the case of the perturbation corresponds to the cosmological horizon. Thus the maximal globally hyperbolic extension of the data on the $s = 0$-hypersurface is extendible.

The case $H_*^{(0)} < 0$ can be discussed similarly. The relevant region in Fig. 2 is now marked as region B, but note that we have to go backwards with respect to the Schwarzschild-de-Sitter time due to the definition of $\hat{s}$ in Eq. (4.5). For the perturbation $(\tilde{M}, \tilde{g})$ the initial expansion of the $\mathbb{S}^2$-factor is now negative and the result is that the future Nariai asymptotics of (M, g) turn into a curvature singularity for the perturbation $(\tilde{M}, \tilde{g})$. The singularity is of cigar type according to the classification in [28], because, although the volume of the spatial surfaces shrinks to zero, the volume of the $\mathbb{S}^1$-factor becomes infinite at the singularity. In the past, both (M, g) and $(\tilde{M}, \tilde{g})$ develop a Cauchy horizon. By means of similar arguments, the cases $\sigma_0 \geq 0$ can also be studied.

5 Summary and outlook

This paper is devoted to the discussion of the cosmic no-hair conjecture and its relation to the Nariai solutions. Our discussion suggests that Nariai asymptotics is a non-generic phenomenon for solutions of Einstein's field equations in vacuum with a positive cosmological constant. We have described the relation of cosmic no-hair and conformal boundaries first, and then analyzed the global properties of Nariai solutions in order to prove the non-existence of smooth conformal boundaries. We point out that the analysis of conformal boundaries serves as a good general diagnostic tool in order to find out about the cosmic no-hair picture, as we also see in [4]. Then, we have explained the instability of the Nariai solutions in the spatially homogeneous case in a precise manner.

Beyond the problem of cosmic no-hair, the particular realization of the instability of the Nariai solution in terms of the sign of the quantity $H_*^{(0)}$ makes it tempting to study the following question in the spatially inhomogeneous case. Can we exploit the instability of the Nariai solutions, which we understand in the spatially homogeneous case, and construct arbitrarily complicated cosmological black hole solutions by making $H_*^{(0)}$ spatially dependent on the initial hypersurface? In principle, we are interested in studying generic inhomogeneous perturbations of the Nariai solutions by prescribing a function $H_*^{(0)}$ without any symmetries. However, this seems hopeless in practice. A

systematic approach would be to reduce the symmetry step by step. The first systematic step for the study of inhomogeneous perturbations is the spherically symmetric case; i.e. we give up the homogeneity along the $\mathbb{S}^1$-factor of the spatial manifold, but keep all the symmetries of the $\mathbb{S}^2$-factor. Much is known about this case locally, for instance by the Birkhoff theorem in [27]. But to our knowledge, there are no rigorous results of a global nature which are relevant for our questions here. Nevertheless, Bousso in [6] claims that cosmological black holes can be constructed as above in the spherically symmetric case.

We have decided not to proceed with the spherically symmetric case in our second paper [4], but rather with Gowdy symmetry. This symmetry is characterized by the presence of two spatial Killing vector fields. It has turned out that it is very difficult to handle this class analytically, and to our knowledge, no cosmological black hole solutions in this class have been found before. This is one motivation for us to proceed with this class by means of numerical techniques. We have developed numerical techniques which can also be applied to the $\mathbb{S}^1 \times \mathbb{S}^2$-Gowdy class of solutions in [5]. Our approach and the first steps in the investigation of the instability of Nariai asymptotics and the construction of cosmological black hole solutions within the Gowdy class are presented in [4].

6 Acknowledgments

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